HOMEWORK #9 SOLUTIONS

7.2.9. We can write the quadratic form as

$$Q(x) = x^T A x$$

where

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

This has characteristic equation

$$(\lambda - 2)(\lambda - 7) = 0$$

So A has eigenvalues $2, 7$ which are both positive therefore Q is positive definite.

We have

$$A - 2I = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

so $\mathbf{u_1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a unit eigenvector. Similarly we have

$$A - 7I = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix}$$

with unit eigenvector $\mathbf{u_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. As the eigenvectors $\mathbf{u_1}, \mathbf{u_2}$ belong to distinct eigenvalues they are orthogonal so we can construct an orthonormal matrix

$$P = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

such that

$$A = PDP^T$$

where

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

Taking the change of coordinates $x = Py$ the quadratic form becomes

$$\tilde{Q}(y) = 2y_1^2 + 7y_2^2$$

4.2.15. The auxilary equation is

$$r^2 + 2r + 1 = 0$$

with repeated root $r = 1$ so the general form of the solution to the homogeneous equation is given by

$$y(t) = Ae^{-t} + Bte^{-t}$$

From the initial condition $y(0) = 1$ we get $A = 1$. From the initial condition $y'(0) = -3$ we get $B = -2$. So the solution to the IVP is given by

$$y(t) = e^{-t} - 2te^{-t}$$

4.6.3. We take as out two linearly independent solutions to the homogeneous equation

$$y_1(t) = e^{-t} \qquad y_2(t) = e^{2t}$$

with Wronskian

$$W(t) = 3e^t$$

The variation of parameters formulae then give us

$$A'(t) = -\frac{2e^{3t}e^{2t}}{6e^t} \qquad B'(t) = \frac{2e^{3t}e^{-t}}{6e^t}$$

Integrating these we get

$$A(t) = -\tfrac{1}{12}e^{4t} + C_1 \qquad B(t) = \tfrac{1}{3}e^t + C_2$$

so the general solution is given by

$$y(t) = C_1e^{-t} + C_2e^{2t} + \tfrac{1}{4}e^{3t}$$